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Error estimates for matrix-valued radial basis function interpolation

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Abstract

We introduce a class of matrix-valued radial basis functions (RBFs) of compact support that can be customized, e.g. chosen to be divergence-free. We then derive and discuss error estimates for interpolants and derivatives based on these matrix-valued RBFs.

Keywords: Error estimates; Interpolation; Radial basis functions; Customized; Divergence free

1. Introduction

Several applications of radial basis functions (RBFs) require that specific physical properties of the data are reflected by the interpolant. For example, if the data comes from the velocity field of the flow of an incompressible fluid, it is desirable that the interpolant be *divergence-free* (i.e. the vector field $\mathbf{v}(x)$ fulfills $\nabla \cdot \mathbf{v} \equiv 0$). Since divergence-free scalar-valued interpolants do not exist, Narcowich and Ward [8] addressed this problem by constructing matrix-valued RBFs that give rise to divergence-free interpolants. However, since these functions are generated by smooth RBFs with *unbounded support*, the corresponding interpolants are not particularly well suited for computation. In this paper we exhibit RBF interpolants that are not only divergence free, but are also much more efficient for computational purposes because of their compact support.

In the next section we construct the new RBF interpolants. The error estimates will be obtained in the context of a *reproducing kernel Hilbert space* or *native space* tailored to the new type of

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RBF. The classic, scalar-valued type of native space can be extended, but some more work is required. This is the topic of Section 2. We then proceed to derive error estimates for the new RBFs. The final estimates will be of the form

$$\|D^\alpha(f - s_{f,X})\|_\infty \leqslant Ch^{l-|\alpha|}$$

for some power $l > 0$, where D^α is a differential operator, X describes the set of scattered data sites, $s_{f,X}$ is the interpolant of f, based on the set X and the matrix-valued RBF Φ, h is the *mesh norm*, and C is a constant depending on f and Φ, but which is independent of the mesh norm. *No* estimates of *any* type for the divergence-free RBFs have been previously derived. First error estimates are obtained in Section 3, and the resultant power function is bounded in Section 4. Estimates for one of the two major quantities arising in these bounds will be obtained in Section 5, using the method of norming sets. Section 6 yields error estimates on the cube. Final error estimates on Ω, a compact subset of $\mathbb{R}^s$, are obtained in Section 7. Some examples and remarks conclude the paper.

2. Notation and preliminaries

A general form of a divergence-free, $s \times s$ matrix-valued RBF Φ is given by the expression

$$\Phi(x) = \{\nabla\nabla^T - \Delta I\}\psi(x), \tag{1}$$

where ψ is a generating scalar-valued RBF, and $\nabla\nabla^T - \Delta I$ is the matrix-valued differential operator consisting of the gradient ∇, the Laplacian operator Δ, and the s-dimensional identity matrix I. Expanding on the method detailed in [8], where ψ was taken to be the smooth *Gaussian* $e^{-t\|\cdot\|^2}$ for $t > 0$, we use polynomials with a finite number of continuous derivatives instead. We employ the recently developed C^{2k}-*Wendland functions* [10] of compact support as our choice for ψ. In addition to being symmetric and divergence-free, the resulting matrix-valued RBF is also positive definite and compactly supported. Formal proofs of these facts are given in [5, Theorem 3.2, Lemma 3.5].

Suppose K is a positive integer. Let $\mathcal{E}_K$ denote the set of all functions $f : \mathbb{R}^s \to \mathbb{C}^s$ such that each component of f belongs to $C^K(\mathbb{R}^s)$. Let $\mathcal{E}'_K$ be the set of all compactly supported $\mathbb{C}^s$-valued distributions, i.e. $\lambda = (\lambda(1), \ldots, \lambda(s))^T$, with $\lambda(j) \in (C^K(\mathbb{R}^s))'$ denoting the jth coordinate of λ for $j = 1, \ldots, s$. If $\lambda \in \mathcal{E}'_K$, then let $\lambda^* := \bar{\lambda}^T$ be the conjugate transpose of λ. The *linear functional* corresponding to the distribution λ acts on $f \in \mathcal{E}_K$ via $(\lambda^*, f) = \sum_{j=1}^{s}(\bar{\lambda}(j), f(j))$.

Suppose ν is a positive integer. Let $\mathbf{B} := \{B_j(x)\}_{j=1}^{\nu}$ be a set of $\mathbb{C}^s$-valued polynomials defined on $\mathbb{R}^s$. A subset $\mathcal{G}_\mathbf{B}$ of $\mathcal{E}_K$ is said to be $\mathbf{B}$-*admissible* if $B_j(\nabla)^* f \equiv 0$ for every $f \in \mathcal{G}_\mathbf{B}$ and every $1 \leqslant j \leqslant \nu$. Observe that if $\nu = 1$, $s \geqslant 2$, and $B_1(x) = x$, then $\mathcal{G}_\mathbf{B} = \mathcal{G}_{\text{div}}$, the admissible space of *divergence-free* vector-valued functions. Although this paper will focus almost exclusively on $\mathcal{G}_{\text{div}}$, the reader will observe that our methods and results allow suitable extensions to other situations as well.

Assume that we are given data $(\lambda_j^*, f) = d_j$, where each d_j, $1 \leqslant j \leqslant N$, is a scalar and $\Lambda := \{\lambda_j\}_{j=1}^N$ is a linearly independent subset of $\mathcal{E}'_K$. To avoid redundant data, we require that $\{(\lambda_j^*, f)\}_{j=1}^N$ be linearly independent for every $f \in \mathcal{G}_\mathbf{B}$; we then say Λ is $\mathcal{G}_\mathbf{B}$-*linearly independent*.

In order to deal with interpolation problems requiring polynomial reproduction, we introduce the following subspaces: given a positive integer m, we let $\mathbb{P}_m^{s \mapsto s}$ denote the collection of all $p : \mathbb{R}^s \to \mathbb{C}^s$ such that each component of p is an s-variate polynomial whose (total) degree is at most $m-1$. Define $\mathcal{P}_m := \mathbb{P}_m^{s\mapsto s} \cap \mathcal{G}_{\mathbf{B}}$,

$$\mathcal{E}'_{K,m}(\mathcal{G}_{\mathbf{B}}) := \{\lambda \in \mathcal{E}'_K : (\lambda^*, p) = 0 \text{ for all } p \in \mathcal{P}_m\}, \quad m \geqslant 1, \tag{2}$$

and $\mathcal{E}'_{K,0}(\mathcal{G}_{\mathbf{B}}) := \mathcal{E}'_K$. Suppose λ and $\tilde{\lambda}$ belong to $\mathcal{E}'_K$. Let Φ be an $s \times s$ matrix each of whose entries is a function in $C^{2K}(\mathbb{R}^s)$; let Φ_j denote the jth column of Φ. Define $(\lambda^* \otimes \tilde{\lambda}, \Phi) := (\lambda^*, \sum_{j=1}^{s} \Phi_j * \tilde{\lambda}(j))$. Assume further that all columns of Φ belong to $\mathcal{G}_{\mathbf{B}}$, and that $\Phi(x)^* = \Phi(-x)$ for all $x \in \mathbb{R}^s$. We say that Φ is an *(order-m) $\mathcal{G}_{\mathbf{B}}$-conditionally positive definite* ($\mathcal{G}_{\mathbf{B}}$-CPD) *function* if

$$(\lambda^* \otimes \lambda, \Phi) \geqslant 0 \quad \text{for all } \lambda \in \mathcal{E}'_{K,m}(\mathcal{G}_{\mathbf{B}}). \tag{3}$$

If equality in (3) implies that $(\lambda^*, g) = 0$ for all $g \in \mathcal{G}_{\mathbf{B}}$, we say that Φ is *strictly* $\mathcal{G}_{\mathbf{B}}$-CPD. When $m = 0$, we say that Φ is *strictly positive definite* (SPD). Given a strictly $\mathcal{G}_{\mathbf{B}}$-CPD matrix-valued function Φ, we define an inner product on $\mathcal{E}'_{K,m}(\mathcal{G}_{\mathbf{B}})$ as follows: for $\lambda, \tilde{\lambda} \in \mathcal{E}'_{K,m}(\mathcal{G}_{\mathbf{B}})$, let $\langle \lambda, \tilde{\lambda} \rangle_\Phi := (\tilde{\lambda}^* \otimes \lambda, \Phi)$. The norm induced by this inner product is denoted by $\|\cdot\|_\Phi$. Suppose now that $f \in \mathcal{E}_K$. We define

$$|f|_\Phi := \sup_{\substack{\lambda \in \mathcal{E}'_{K,m}(\mathcal{G}_{\mathbf{B}}) \\ \|\lambda\|_\Phi = 1}} |(\lambda^*, f)|, \tag{4}$$

and the *native space* of Φ to be $\mathcal{N}_\Phi := \{f \in \mathcal{E}_K : |f|_\Phi < \infty\}$.

Note that we required all elements of Φ to belong to $C^{2K}(\mathbb{R}^s)$, with $2K$ an even integer, in order to guarantee that the expression $(\lambda^* \otimes \lambda, \Phi)$ in (3) is well defined for $\lambda \in \mathcal{E}'_K$. We are now ready to formulate our main problem [1,2,7,8].

Problem 1 (*Generalized Hermite Interpolation Problem*). Assume that Φ is a strictly order-m $\mathcal{G}_{\mathbf{B}}$-CPD, $s \times s$ matrix-valued function. Let $\Lambda = \{\lambda_j\}_{j=1}^N$ be a $\mathcal{G}_{\mathbf{B}}$-linearly independent set of distributions in $\mathcal{E}'_K$, and let f be a function in $\mathcal{E}_K$. Given the data $d_j = (\lambda_j^*, f)$, $1 \leqslant j \leqslant N$, find $\lambda \in \operatorname{span}\{\Lambda\} \cap \mathcal{E}'_{K,m}(\mathcal{G}_{\mathbf{B}})$ and $p \in \mathcal{P}_m$ such that $\Phi * \lambda \in \mathcal{G}_{\mathbf{B}}$ and $s_{f,X} = \Phi * \lambda + p$ satisfies: if f is in $\mathcal{P}_m$, then $s_{f,X} = p = f$, and further

$$(\lambda_j^*, s_{f,X}) = d_j \quad \text{for } 1 \leqslant j \leqslant N. \tag{5}$$

We assume that the function f generating the data belongs to the function class $\mathcal{F}$, defined as:

Definition 2. We say that a function f belongs to $\mathcal{F}$ if (i) $f \in \mathcal{E}_K$, and (ii) f is representable in the form $f = \Phi * \lambda + p$, where $p \in \mathcal{P}_m$ and $\lambda \in \mathcal{E}'_{K,m}(\mathcal{G}_{\mathbf{B}})$.

Note that $|f|_\Phi = \|\lambda\|_\Phi$ holds for any $f \in \mathcal{F}$, and hence $\mathcal{F} \subset \mathcal{N}_\Phi$. Let us point out that if, in Problem 1, m is chosen to be zero and $\lambda_j = v_j \delta_{x_j}$, $v_j \in \mathbb{C}^s$, $x_j \in \mathbb{R}^s$, and δ_{x_j} is a Dirac δ-distribution at x_j for $1 \leqslant j \leqslant N$, then the interpolant $s_{f,X}$ takes the well-known form $\sum_{j=1}^N \Phi(\cdot - x_j) v_j$. The last result in this section guarantees that Problem 1 always has a unique solution. Its proof, which is a direct modification of [8, Theorem 2.3], is omitted.

Theorem 3. *If the dimensions of* $\operatorname{span}\{\Lambda\} \setminus \mathcal{E}'_{K,m}(\mathcal{G}_{\mathbf{B}})$ *and* $\mathcal{P}_m$ *agree, then Problem* 1 *has a unique solution.*

We now turn to the task of establishing error bounds for the interpolants described in Problem 1. The corresponding issue for scalar-valued RBFs has been addressed in [6,9,12]. Here we shall extend the method from [9] and obtain comparable estimates for matrix-valued RBF interpolants possessing additional constraints.

3. Initial error estimates

For the remainder of the paper, we assume the following: Let $\mathcal{G}_{\mathbf{B}} = \mathcal{G}_{\mathrm{div}} =: \mathcal{G}$, further, let $K = k/2$ for k an even integer, and let Φ be an $s \times s$ matrix-valued, strictly order-m $\mathcal{G}$-CPD function with its components being in $C^k(\mathbb{R}^s)$, and let $f \in \mathcal{F}$. Further suppose that $\Lambda = \{\lambda_j\}_{j=1}^N$ is as in Problem 1. Our first estimate is contained in the following proposition, which serves to bound the interpolation error in the terms of the quantity $|f|_\Phi$ and the *power function*, defined as

$$P_{\Phi,\Lambda}^{\eta} := \min \left\| \eta - \sum_{j=1}^{N} c_j \lambda_j \right\|_\Phi ,$$

where the minimum is taken over all constants $c_1, \ldots, c_N$ satisfying the constraint $(\eta^*, p) = \sum_{j=1}^N c_j (\lambda_j^*, p)$ for all $p \in \mathcal{P}_m$. Note that $P_{\Phi,\Lambda}^{\eta}$ is *independent* of *f*. We remark that *pointwise* error estimates are obtained for the choice $\eta = v\delta_x$, where $v \in \mathbb{C}^s$ is some vector and $x \in \mathbb{R}^s$. In this case one finds that $(\eta^*, f - s_{f,X}) = v^*(f - s_{f,X})(x)$.

Proposition 4. *Assume that the hypothesis of Theorem* 3 *is in place and that the above assumptions hold. If* $\eta \in \mathcal{E}_K'$, *then*

$$|(\eta^*, f - s_{f,X})| \leqslant |f|_\Phi P_{\Phi,\Lambda}^{\eta}. \tag{6}$$

Proof. By Theorem 3 there always exists a solution $s_{f,X} = \Phi * \lambda + q$ with $\lambda \in \operatorname{span}\{\Lambda\} \cap \mathcal{E}_{K,m}'(\mathcal{G})$ satisfying $(\lambda_j^*, f - s_{f,X}) = 0$, $1 \leqslant j \leqslant N$, and $q \in \mathcal{P}_m$, the set of all *s*-component polynomials *p* of degree $m-1$ for which $\nabla \cdot p \equiv 0$. From this and the assumption $f = \Phi * \tilde{\lambda} + p$ it immediately follows that $0 = (\lambda^*, f - s_{f,X}) = \langle \tilde{\lambda} - \lambda, \lambda \rangle_\Phi$, and hence $\|\tilde{\lambda}\|_\Phi^2 = \|\tilde{\lambda} - \lambda\|_\Phi^2 + \|\lambda\|_\Phi^2$. Since the interpolation process reproduces $\mathcal{P}_m$, the set $\Lambda^*|_{\mathcal{P}_m} := \{\lambda_j^*\}_{j=1}^N$—if restricted to elements of $\mathcal{P}_m$—spans the dual of $\mathcal{P}_m$. So there exist constants $c_1, \ldots, c_N$ such that $\sum\limits_{j=1}^{N} c_j (\lambda_j^*, p) = (\eta^*, p)$ for all $p \in \mathcal{P}_m$. Thus, $\eta - \sum\limits_{j=1}^{N} c_j \lambda_j$ is in $\mathcal{E}_{K,m}'(\mathcal{G})$. Hence, we can derive

$$|(\eta^*, f - s_{f,X})| = \left| \left(\left(\eta - \sum_{j=1}^{N} c_j \lambda_j \right)^*, f - s_{f,X} \right) \right| \leqslant \|\tilde{\lambda}\|_\Phi \left\| \eta - \sum_{j=1}^{N} c_j \lambda_j \right\|_\Phi . \tag{7}$$

If in (7), we now replace $\|\tilde{\lambda}\|_\Phi$ by the function norm in (4), and take the minimum of (7) over all c_j's satisfying the property $(\eta^*, p) = \sum_{j=1}^N c_j (\lambda_j^*, p)$ for all $p \in \mathcal{P}_m$, we obtain the desired estimate (6). $\square$

The error bound (6) is of limited value unless the resultant power function can be estimated in a useful manner. The remainder of the paper will be devoted to this task.

4. Estimates on the power function

In this section we will derive upper bounds for the power function based on divergence-free RBFs Φ. From Proposition 4, we see that $(P^{\eta}_{\Phi,\Lambda})^2$ is the minimum of the quadratic form

$$Q_\Phi(c_1,\ldots,c_N) := \left(\left(\eta - \sum_{j=1}^{N} c_j\lambda_j\right)^* \otimes \left(\eta - \sum_{j=1}^{N} c_j\lambda_j\right), \Phi\right), \tag{8}$$

with $\sum_{j=1}^{N} c_j(\lambda_j^*, p) = (\eta^*, p)$ for all $p \in \mathcal{P}_m$. In order to bound (8), we will first approximate Φ component wise by a function $\Phi_{\mathcal{P}_m}$ in $\mathcal{P}_m \otimes \mathcal{P}_m^*$. We then estimate Q_Φ by bounding $Q_{\Phi-\Phi_{\mathcal{P}_m}}$ using Taylor residuals. The next proposition establishes that the two quadratic forms agree and also gives a first upper bound on $P^{\eta}_{\Phi,\Lambda}$.

Proposition 5. *Let $\Phi_{\mathcal{P}_m}$ be an $s \times s$ matrix-valued function whose components are in $\mathcal{P}_m \otimes \mathcal{P}_m^*$, and let the c_j's satisfy the constraint $(\eta^*, p) = \sum_{j=1}^{N} c_j(\lambda_j^*, p)$ for all $p \in \mathcal{P}_m$. Then $Q_\Phi = Q_{\Phi-\Phi_{\mathcal{P}_m}}$ and*

$$(P^{\eta}_{\Phi,\Lambda})^2 \leqslant Q_{\Phi-\Phi_{\mathcal{P}_m}}(c_1,\ldots,c_N).$$

Proof. Let $\tilde{\lambda} = \eta - \sum_{j=1}^{N} c_j\lambda_j$. Then $\tilde{\lambda} \in \mathcal{E}'_{K,m}(\mathcal{G})$. Hence, $Q_\Phi = (\tilde{\lambda}^* \otimes \tilde{\lambda}, \Phi)$ by (8). If $\{p_j\}_{j=1}^{M}$ is a basis for $\mathcal{P}_m$, we write $\Phi_{\mathcal{P}_m} = \sum_{j,k=1}^{M} b_{j,k}\, p_j \otimes p_k^*$, where $b_{j,k}$ and $p_j \otimes p_k^*$ are matrices for all j, k. Thus,

$$Q_{\Phi-\Phi_{\mathcal{P}_m}} = (\tilde{\lambda}^* \otimes \tilde{\lambda}, \Phi) - \underbrace{\sum_{j,k=1}^{M} (\tilde{\lambda}^* \otimes \tilde{\lambda}, b_{j,k}\, p_j \otimes p_k^*)}_{=0} = Q_\Phi.$$

By definition, $(P^{\eta}_{\Phi,\Lambda})^2$ is the minimum of Q_Φ, which completes the proof. □

Remark 6. The question naturally arises how functions in $\mathcal{P}_m \otimes \mathcal{P}_m^*$ can be used to locally approximate Φ component wise, with all columns of $\Phi_{\mathcal{P}_m}$ belonging to $\mathcal{G}$; the space $\mathcal{P}_m$ might seem rather small. First, note that one can always enlarge $\mathcal{P}_m$ to consist not only of the functions we wanted our original interpolation problem to reproduce, but of many more. This follows because, given a space $\mathcal{P}$ containing $\mathcal{P}_m$, Φ being a strictly $\mathcal{G}$-CPD function for $\mathcal{P}$ implies that Φ is a strictly $\mathcal{G}$-CPD function for $\mathcal{P}_m$. Second, once $\mathcal{P}_m$ is enlarged in the desired way, Lemma 7 assures that all columns of $\Phi_{\mathcal{P}_m}$ still belong to $\mathcal{G}$.

Lemma 7. *Let $\Phi_{\mathcal{P}_m} = (\Phi_{\mathcal{P}_m\,\rho,\sigma})_{1\leqslant\rho,\sigma\leqslant s}$ be the matrix with its components being the Taylor polynomials of degree $k-1$ at a neighborhood of the origin for the corresponding components of $\Phi = (\Phi_{\rho,\sigma})_{1\leqslant\rho,\sigma\leqslant s}$. Then all columns of $\Phi_{\mathcal{P}_m}$ are in $\mathcal{G}$.*

Proof. Let $\Phi_{\mathcal{P}_m\rho,\sigma}(x) = \sum_{|\beta|<k} D^{\beta}\Phi_{\rho,\sigma}(0)x^{\beta}/\beta!$ for $1 \leqslant \rho, \sigma \leqslant s$. We will now prove that the columns $\Phi_{\mathcal{P}_m,1},\ldots,\Phi_{\mathcal{P}_m,s}$ are in the admissible space $\mathcal{G}$, i.e. that they fulfill $\nabla \cdot \Phi_{\mathcal{P}_m,i} \equiv 0$ for

$1 \leqslant i \leqslant s$. We obtain

$$\partial_{x(i)} \Phi_{\mathcal{P}_m \rho,\sigma}(x) = \sum_{|\beta|<k-1,\, \beta_i \geqslant 1} D^\beta \Phi_{\rho,\sigma}(0) \frac{x_1^{\beta_1} \cdots x_i^{\beta_i - 1} \cdots x_s^{\beta_s}}{\beta_1! \cdots (\beta_i - 1)! \cdots \beta_s!},$$

and hence $\partial_{x(i)} \Phi_{\mathcal{P}_m \rho,\sigma}(x) = \sum_{|\tilde{\beta}|<k-1} D^{\tilde{\beta}}(\partial_{x(i)} \Phi_{\rho,\sigma})(0) x^{\tilde{\beta}} / \tilde{\beta}!$, where we set $\tilde{\beta} := \beta|_{\beta_i \to \beta_i - 1}$. Thus, partial derivatives turn out to work on the coefficients. For general $D^\alpha \Phi_{\rho,\sigma}$, the same idea applies, since any $\partial_{x(i)} \partial_{x(j)}$ can be applied one at a time. We hence obtain $\nabla \cdot \Phi_{\mathcal{P}_m, i}(x) = \sum_{|\tilde{\beta}|<\tilde{k}} D^{\tilde{\beta}}(\nabla \cdot \Phi_i)(0) x^{\tilde{\beta}} / \tilde{\beta}! = 0$, for any column $\Phi_{\mathcal{P}_m, i}$, $1 \leqslant i \leqslant s$, of $\Phi_{\mathcal{P}_m}$ since the derivatives $D^{\tilde{\beta}}$ and ∇ are commutative, where $\tilde{\beta}, \tilde{k}$ are obtained in the above-described matter. □

In [8, Lemma 2.2] it was shown that, if $\Phi(x)^* = \Phi(-x)$, then Φ is a conjugate symmetric matrix-valued function. We now require that Φ fulfill this assumption. Then $\Phi_{\mathcal{P}_m}$ is conjugate symmetric as well. Eq. (8) now yields

$$Q_{\Phi - \Phi_{\mathcal{P}_m}} = (\eta^* \otimes \eta, \Phi - \Phi_{\mathcal{P}_m}) - 2\Re \left\{ \sum_{j=1}^N c_j \left(\eta^* \otimes \lambda_j, \Phi - \Phi_{\mathcal{P}_m} \right) \right\} + \sum_{j,k=1}^N c_j c_k^* (\lambda_k^* \otimes \lambda_j, \Phi - \Phi_{\mathcal{P}_m}). \tag{9}$$

Let $\Delta_0 := |(\eta^* \otimes \eta, \Phi - \Phi_{\mathcal{P}_m})|$, $\Delta_1 := \max_j |(\eta^* \otimes \lambda_j, \Phi - \Phi_{\mathcal{P}_m})|$, and $\Delta_2 := \max_{j,k} |(\lambda_k^* \otimes \lambda_j, \Phi - \Phi_{\mathcal{P}_m})|$. Eq. (9), together with Proposition 5, yields the following estimate for the power function:

Theorem 8. *Let $\Phi_{\mathcal{P}_m}$ be a conjugate symmetric matrix-valued function whose components are in $\mathcal{P}_m$. For any $c = (c_j)_{j=1}^N$ satisfying $(\eta^*, p) = \sum_{j=1}^N c_j (\lambda_j^*, p)$ for all $p \in \mathcal{P}_m$, we have the following upper bound on the power function*:

$$(P_{\Phi,\Lambda}^\eta)^2 \leqslant \Delta_0 + 2\|c\|_1 \Delta_1 + \|c\|_1^2 \Delta_2. \tag{10}$$

5. First upper bounds for $\|c\|$

Our next goal is to obtain bounds on $\|c\|_1$ in (10) satisfying the constraint $(\eta^*, p) = \sum_{j=1}^N c_j (\lambda_j^*, p)$ for all $p \in \mathcal{P}_m$. We will apply the technique of *norming sets* [3,4]. The following proposition provides an upper bound for $\|c\|_1$ which holds uniformly on any compact subset Ω of $\mathbb{R}^s$. We define $T : \mathcal{P}_m \to \mathbb{R}^N$ by $T(p) := ((\lambda_j^*, p))_{j=1}^N$ and $\|T^{-1}\| := \sup_{p \in \mathcal{P}_m,\, p \neq 0} \frac{\|p\|_{\mathcal{P}_m}}{\|T(p)\|_\infty}$.

Proposition 9. *There exist coefficients $c = (c_j)_{j=1}^N$ such that for all $p \in \mathcal{P}_m$, the condition $(\eta^*, p) = \sum_{j=1}^N c_j (\lambda_j^*, p)$ holds. Furthermore, $\|c\|_1 \leqslant \|\eta|_{\mathcal{P}_m}\|_{\mathcal{P}_m^*} \|T^{-1}\|$.*

Proof. Adopting the definition of norming sets [9, Definition 3.3, Proposition 3.4] gives the necessary assumptions for a norming set. The desired result now is obtained following the proof of [9, Corollary 3.5], where V is chosen to be $\mathcal{P}_m$ and the injectivity of T follows by Theorem 3. □

6. Error estimates on a cube

In this section we will derive error estimates for closed cubes contained in Ω, a compact subset of $\mathbb{R}^s$. We will first state necessary assumptions that hold for the remainder of the paper. We will then bound the quantity $\|c\|_1$ on certain cubes and also derive estimates for the Δ_j's. Combining this will give computable upper bounds for the power function $P^{\eta}_{\Phi,\Lambda}$ in (10).

Assumption 10. Let Φ be in $C^k_\nu(\mathbb{R}^s)$, with k even, i.e. Φ has k derivatives that are Hölder continuous at the origin. Further, let the set Λ comprise vector-valued Dirac δ-distributions $\lambda_j = v_j \delta_{x_j}$, with $v_j \in \mathbb{R}^s$ for $1 \leqslant j \leqslant N$, based on data sites $X = \{x_j\}_{j=1}^N \subset \Omega$ in $\mathbb{R}^s$. Let η be given by $\eta = (-1)^{|\alpha|} v D^\alpha \delta_x$, where $x \in \Omega$, $v \in \mathbb{R}^s$, and $|\alpha| \leqslant k/2$. We define the *mesh norm*, or Hausdorff distance, for Ω with respect to X to be

$$h := \sup_{y \in \Omega} \min_{x_j \in X} \|y - x_j\|_2. \tag{11}$$

We assume that x, via η, is contained in a closed cube $W(w, \delta) := \{x \in \mathbb{R}^s : \|x - w\|_\infty \leqslant \delta\} \subset \Omega$. We abbreviate $W := W(w, \delta)$. Let $\delta > h$, which guarantees that $Y := X \cap W$ is non-empty.

The following proposition now yields an upper bound on W for $\|c\|_1$. We let $h_{Y,W} := \sup_{z \in W} \min_{x_j \in Y} \|z - x_j\|_2$ be the *mesh norm for W with respect to Y*.

Proposition 11. *If $h_{Y,W} \leqslant \delta/[2\sqrt{s}(m-1)^2]$ holds, then for every $x \in W = W(w, \delta)$, there exist coefficients $c = (c_j)_{j=1}^N$ such that $(\eta^*, p) = \sum_{j=1}^N c_j (\lambda_j^*, p)$ for all $p \in \mathcal{P}_m$ and $\|c\|_1 \leqslant 2\|v\|_1 [(m-1)^2/\delta]^{|\alpha|}$.*

Proof. We mainly follow the proof of [9, Lemma 6.1] with a few exceptions based on the vector-valued nature of $p \in \mathcal{P}_m$ which we will identify here. Define a norm on $\mathcal{P}_m$, $\|p\|_{\infty,W} := \sup_{x \in W} \max_{1 \leqslant i \leqslant s} |p_i(x)|$, where p_i denotes the ith component of $p \in \mathcal{P}_m$. Similar to [9, Lemma 6.1] we obtain the following estimate for some $\xi \in \mathbb{R}^s$: since $\|p\|_{\infty,W} = |p_l(z)|$ for some $z \in W$ and $l \in \{1, \ldots, s\}$, applying the mean value theorem gives

$$|p_l(y)| \geqslant \|p\|_{\infty,W} - \left| \sum_{j=1}^s \frac{\partial p_l}{\partial x(j)} (\xi)(z(j) - y(j)) \right|, \tag{12}$$

for some $\xi \in W$, where y is the closest point in Y to z, and $x(j)$ is the jth component of x, etc. Using the following estimate derived in [9]:

$$\|D^\alpha p\|_{\infty,W} \leqslant \left(\frac{(m-1)^2}{\delta} \right)^{|\alpha|} \|p\|_{\infty,W}, \tag{13}$$

we find that $\|p\|_{\infty,Y} \geqslant |p_l(y)| \geqslant 1/2 \|p\|_{\infty,W}$, where we defined $\|p\|_{\infty,Y}$ to be $\sup_{y \in Y} \max_{1 \leqslant i \leqslant s} |p_i(y)|$. But this is equivalent to the—in Section 5 defined—operator $T(p) = (v_j^* p(x_j))_{j=1}^N$ being injective, with $\|T^{-1}\| \leqslant 2$. A short calculation now gives $|(\eta^*, p)| \leqslant \|v\|_1 \sup_{1 \leqslant j \leqslant s} |D^\alpha p_j(x)|$, and hence $\|(\eta^*, p)\|_{\mathcal{P}_m^*} \leqslant \|v\|_1 [(m-1)^2/\delta]^{|\alpha|}$ for all $p \in \mathcal{P}_m$, via (13). Combining this and Proposition 9 yields the result. $\square$

Note that Remark 6 ensures that if $m = 1$ in Proposition 11, one can enlarge m such that $\delta/[2\sqrt{s}(m-1)^2]$ is well defined. The following theorem bounds the power function on W via (10). It shows the specific dependencies of all variables; a more general formulation is given in Theorem 14. The next result combines Proposition 11 with estimates for the Δ_j's derived in its proof. Let $\Lambda_Y \subset \Lambda$ be the set of Dirac δ-distributions at points in Y.

Theorem 12. *Let Assumption* 10 *be fulfilled. If the mesh norm satisfies that* $h_{Y,W} \leqslant \delta/[2\sqrt{s}(m-1)^2]$, *then the power function* $P^{\eta}_{\Phi,\Lambda_Y}$ *may be estimated as follows on* W: *define* $r_{s,m} := 2\sqrt{s}(m-1)^2$, $\tilde{v} := \max_{x_j\in Y} \|v_j\|_1$, *and* $M^{\Phi}_{k,v} := \max_{\substack{1\leqslant\rho,\sigma\leqslant s\\ |\beta|=k}} \|D^{\beta}\Phi_{\rho,\sigma}\|_{C_v}$, *then*

$$(P^{\eta}_{\Phi,\Lambda_Y})^2 \leqslant 4\,(2\sqrt{s}\,\delta)^{k+v-2|\alpha|}\, M^{\Phi}_{k,v}\, r^{|\alpha|}_{s,m}\, \tilde{v}\, \|v\|_1^2 \left(\frac{s^{(k-|\alpha|)/2}}{(k-|\alpha|)!} + r^{|\alpha|}_{s,m} \frac{s^{k/2}}{k!}\, v \right). \tag{14}$$

Proof. Let $\phi_{\mathcal{P}_m}$ be the Taylor polynomial of degree $k-1$ for a scalar-valued function $\phi \in C^k(\mathbb{R}^s)$. If necessary, enlarge $\mathcal{P}_m$ such that it consists of polynomials of total degree not less than $k-1$. The following inequality was established in [9]:

$$|D^{\gamma}(\phi - \phi_{\mathcal{P}_m})(t)| \leqslant C\, \|t\|_2^{k+v-|\gamma|}, \quad |\gamma| \leqslant k, \tag{15}$$

with $C := \frac{s^{(k-|\gamma|)/2}\tilde{M}^{\phi}_{k,v}}{(k-|\gamma|)!}$, $\tilde{M}^{\phi}_{k,v} := \max_{|\beta|=k} \|D^{\beta}\phi\|_{C_v}$, and $\|t\|$ being sufficiently small. A short calculation gives

$$\Delta_0 \leqslant \|v\|_1^2 \max_{1\leqslant\rho,\sigma\leqslant s} |D^{2\alpha}(\Phi_{\rho,\sigma} - \Phi_{\mathcal{P}_m\rho,\sigma})(0)|.$$

Applying (15) componentwise with $\phi = \Phi_{\rho,\sigma}$, $\gamma = 2\alpha$, and $t = 0$ now yields that $\Delta_0 = 0$. Applying (15) once again with $\gamma = \alpha$, we see that

$$\begin{aligned} \Delta_1 &= \max_{x_j\in Y} \left| \sum_{\rho,\sigma=1}^{s} \bar{v}(\rho) D^{\alpha}(\Phi_{\rho,\sigma} - \Phi_{\mathcal{P}_m\rho,\sigma})(x - x_j) v_j(\sigma) \right| \\ &\leqslant \|v\|_1 \max_{x_j\in Y} \|v_j\|_1 \frac{s^{(k-|\alpha|)/2}}{(k-|\alpha|)!} M^{\Phi}_{k,v} (2\sqrt{s}\delta)^{k+v-|\alpha|}, \end{aligned} \tag{16}$$

because $\|x - x_j\| \leqslant 2\sqrt{s}\delta$. Similarly,

$$\Delta_2 \leqslant \max_{x_j\in Y} \|v_j\|_1^2 \frac{s^{k/2}}{k!} M^{\Phi}_{k,v}\, (2\sqrt{s}\delta)^{k+v}. \tag{17}$$

Combining Proposition 11, Theorem 8, (16), and (17), with the fact that $\Delta_0 = 0$ yields (14). □

7. Estimates on Ω and examples

We are now in the position to derive uniform error estimates on Ω for matrix-valued RBF interpolants. We will need the following result that relates the mesh norms on the cube and on Ω. For a proof see [9].

Lemma 13. *Given a closed cube* $W = W(w,\delta)$ *such that* $\delta > h$. *Then* $h_{Y,W} \leqslant (1+\sqrt{s})h$.

In Theorem 14 we will use $\|g\|_\infty := \sup_{v\in\mathbb{R}^s,\|v\|_1=1} |v^* g(x)|$. We further set $\mathcal{T}^2 := M^\Phi_{k,v} \left(2\mathcal{R}^{k+v-|\alpha|}\mathcal{C}\frac{s^{(k-|\alpha|)/2}}{(k-|\alpha|)!} + \mathcal{R}^{k+v}\mathcal{C}^2\frac{s^{k/2}}{k!}\right)$, where $\mathcal{R} := 4s(1+\sqrt{s})(m-1)^2$ and $\mathcal{C} := 2^{1-|\alpha|}(s+\sqrt{s})^{-|\alpha|}$.

Theorem 14. *Let Assumption* 10 *be fulfilled. If* $\max_{x_j\in X}\|v_j\|_1 \leqslant 1$, $\|v\|_1 \leqslant 1$, *and* $\delta = 2(\sqrt{s}+s)(m-1)^2 h$, *then*

$$\sup_{x\in W(\omega,\delta)\subset\Omega} \|D^\alpha(f - s_{f,X})(x)\|_\infty \leqslant |f|_\Phi \mathcal{T} h^{\frac{k+v}{2}-|\alpha|}, \tag{18}$$

where $\mathcal{T}$ *and* $M^\Phi_{k,v}$ *are defined as above and in Theorem* 12, *respectively.*

Proof. Let $W = W(w,\delta)$. Since $\delta = 2(\sqrt{s}+s)(m-1)^2 h$ and $h_{Y,W} \leqslant (1+\sqrt{s})h$ by Lemma 13, we get $h_{Y,W} \leqslant \delta/[2\sqrt{s}(m-1)^2]$. Hence, the assumptions for Theorem 12 are fulfilled and (14) holds. Since $W \subset \Omega$, and since $Y \subset X$ implies that $\Lambda_Y \subset \Lambda$, we get $P^\eta_{\Phi,\Lambda} \leqslant P^\eta_{\Phi,\Lambda_Y}$. Replacing Λ_Y by Λ and W by Ω in (14) yields

$$(P^\eta_{\Phi,\Lambda})^2 \leqslant M^\Phi_{k,v}\left(2\mathcal{R}^{k+v-|\alpha|}\mathcal{C}\,\frac{s^{(k-|\alpha|)/2}}{(k-|\alpha|)!} + \mathcal{R}^{k+v}\mathcal{C}^2\,\frac{s^{k/2}}{k!}\right) h^{k+v-2|\alpha|}. \tag{19}$$

Finally, combining (6) with (19), we obtain the desired uniform error estimate on Ω. $\square$

Remark 15. Theorem 14 says that for any x contained in a cube $W \subset \Omega$ of a certain length, the error can be bounded uniformly. Also note that the upper bound (18) consists of three terms, only $|f|_\Phi$ depending on the function f, and only the last quantity depending on the mesh norm h.

A set of matrix-valued RBFs that fulfill the assumptions of our theorems is given in the following proposition.

Proposition 16. *Let* ψ *be a Wendland function in* $C^{2k+2}(\mathbb{R}^s)$. *Let* $\Phi(\mathbf{x}) = \{\nabla\nabla^T - \Delta I\}\psi(\mathbf{x})$ *be the* $s\times s$ *matrix-valued function based on the Wendland function* ψ *with its components in* $C^{2k}(\mathbb{R}^s)$. *Then*

$$\sup_{x\in W(\omega,\delta)\subset\Omega} \|D^\alpha(f - s_{f,X})(x)\|_\infty \leqslant |f|_\Phi C_\alpha h^{k+\frac{1}{2}-|\alpha|}. \tag{20}$$

Proof. In the proof of [11, Theorem 11.17] the following inequality was given:

$$|D^\gamma(\psi - p)(t)| \leqslant C_\gamma\, \|t\|_2^{2k+3-|\gamma|},$$

for $\gamma \leqslant 2k+2$ and $\psi \in C^{2k+2}(\mathbb{R}^s)$ being a Wendland function. Hence, in the matrix-valued case $\Phi = \{\nabla\nabla^T - \Delta I\}\psi(\mathbf{x}) \in C^{2k}(\mathbb{R}^s)$ we obtain componentwise

$$|D^\gamma(\Phi_{\rho,\sigma} - \Phi_{\mathcal{P}_m\rho,\sigma})(t)| \leqslant C_\gamma\, \|t\|_2^{2k+1-|\gamma|} \quad \text{for } |\gamma| \leqslant 2k.$$

Using the last inequality instead of (15) for deriving upper bounds for Δ_0, Δ_1, and Δ_2 then yields the desired result. $\square$

We will conclude the paper with a two-dimensional numerical example where all needed assumptions are present and the numerical results are compared to the theoretical bounds. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the divergence-free function given by $\mathbf{f}(\mathbf{x}) = (u(\mathbf{x}), v(\mathbf{x}))^T$ with $\mathbf{x} = (x, y)^T$, $u(\mathbf{x}) = x^3 - 3xy^2 + y$, and $v(\mathbf{x}) = y^3 - 3x^2y + 2$. We assume that we have given data d_j, for $j = 1, \ldots, 2N$, with $d_j = u(\mathbf{x}_j)$ and $d_{j+N} = v(\mathbf{x}_j)$ at locations $\mathbf{x}_j$, for $j = 1, \ldots, N$, on a regular grid in $\Omega := [-1, 1] \times [-1, 1]$. The spacing between points is given by $dx = dy = 2/(\sqrt{N} - 1)$. The velocity data for $N = 256$ locations is displayed in Fig. 1. Then, the mesh norm for Ω with respect to X is given by $h = \frac{1}{2}\sqrt{dx^2 + dy^2}$. We wish to interpolate the data employing a divergence-free RBF of the form (1). Since $f \in \mathcal{E}_K$, the data can be described by the $\mathcal{G}_{\text{div}}$-linearly independent set of distributions $\Lambda := \{\lambda_j\}_{j=1}^{2N} \in \mathcal{E}'_K$ with $\lambda_j = (1, 0)^T \delta_{\mathbf{x}_j}$ and $\lambda_{j+N} = (0, 1)^T \delta_{\mathbf{x}_j}$, for $j = 1, \ldots, N$. Further, we choose $\Phi(\mathbf{x}) = \{\nabla\nabla^T - \Delta I\}\psi(\mathbf{x}) \in C^2(\mathbb{R}^2)$ with $\psi = \phi_{4,2}$ being the $\in C^4$-Wendland function as interpolating RBF. Then the theoretical approximation result from the previous proposition is valid. Observe that for $\mathbf{x} = (x, y)^T$ and $\Phi(\mathbf{x}) := (\Phi_{i,j}(\mathbf{x}))_{1 \leqslant i,j \leqslant 2}$ we have

$$\begin{aligned}
\Phi_{1,1}(\mathbf{x}) &= -56/3(1 - \|\mathbf{x}\|)_+^4 (5x^2 + 35y^2 - 4\|\mathbf{x}\| - 1),\\
\Phi_{1,2}(\mathbf{x}) &= \Phi_{2,1}(\mathbf{x}) = 560(1 - \|\mathbf{x}\|)_+^4 xy,\\
\Phi_{2,2}(\mathbf{x}) &= -56/3(1 - \|\mathbf{x}\|)_+^4 (35x^2 + 5y^2 - 4\|\mathbf{x}\| - 1).
\end{aligned}$$

For $\lambda \in \operatorname{span}\{\Lambda\}$ the interpolant now takes the form

$$\mathbf{s}_{f,X}(\mathbf{x}) = \Phi * \lambda(\mathbf{x}) = \sum_{j=1}^{N} \alpha_j \begin{pmatrix} \Phi_{1,1}(\mathbf{x} - \mathbf{x}_j) \\ \Phi_{2,1}(\mathbf{x} - \mathbf{x}_j) \end{pmatrix} + \sum_{j=1}^{N} \alpha_{j+N} \begin{pmatrix} \Phi_{1,2}(\mathbf{x} - \mathbf{x}_j) \\ \Phi_{2,2}(\mathbf{x} - \mathbf{x}_j) \end{pmatrix}. \tag{21}$$

We wish to get an error estimate for $\sup_{\mathbf{x} \in W(\omega,\delta) \subset \Omega} \|\mathbf{f}(\mathbf{x}) - \mathbf{s}_{f,X}(\mathbf{x})\|_\infty$ using $\eta = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})^T \delta_{\mathbf{x}}$ and $|\alpha| = 0$. Hereby, let $\mathbf{x}$ be contained in $W(\omega, \delta) \subset \Omega$ with $\delta = \min(1, 2(\sqrt{2} + 2)h)$ and $\omega = 0$ without loss of generality.

We now turn to the numerical results. Let $N = 9, 36, 144, 576, 2304$ be the number of data points. For fixed N, the interpolation error is measured via ERROR_N as follows. For $\text{ERROR}_N := \sup_{\mathbf{x} \in W(\omega,\delta) \cap \tilde{X}} \|\mathbf{f}(\mathbf{x}) - \mathbf{s}_{f,X}(\mathbf{x})\|_\infty$, the interpolant was obtained using the above N data points and the final l_∞-error was calculated on a refined data set $\tilde{X}$ based on a grid of spacing $d\tilde{x} = d\tilde{y} = 16\, dx$. We assume that the interpolation error satisfies

$$\sup_{\mathbf{x} \in W(\omega,\delta) \subset \Omega} \|\mathbf{f}(\mathbf{x}) - \mathbf{s}_{f,X}(\mathbf{x})\|_\infty \leqslant |\mathbf{f}|_\Phi C(\mathbf{s}, k, \alpha, m) h^{t-|\alpha|},$$

for some integer t, space dimension s, continuity constant k of the RBF, derivative index vector α, and for the polynomial order m. For $|\alpha| = 0$ we define

$$\text{ratio}_N := \frac{\text{ERROR}_N}{\text{ERROR}_{4N}} \approx 2^t,$$

applying that quadrupling N leads to dividing h by 2. Table 1 summarizes our findings.

It may be observed that the approximation order t lies between 1.5 and 2.1, depending on the size of the mesh norm h. Proposition 16 (applied with $|\alpha| = 0$ and $k = 1$) yields the value $t = 1.5$. The deviation may be caused by stability-influencing factors of the numerical example, such as the number of interpolation and evaluation points as well as the stability of the inversion algorithm used in MATLAB. Note that a higher value of t leads to a better error estimate since h is smaller

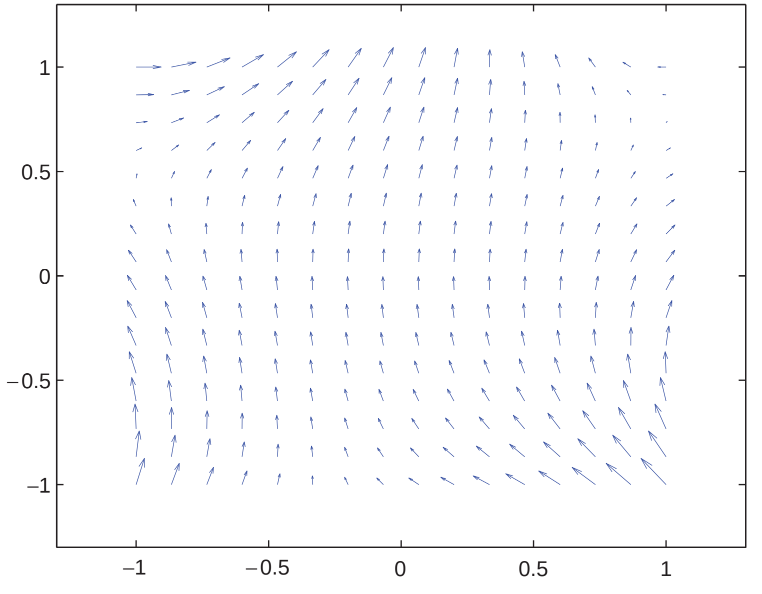

Fig. 1. Numerical example: velocity data for $N = 256$.

Table 1
Errors for the example

N	h	ERROR_N	ratio_N	t
9	0.7071	3.0927	–	–
36	0.2828	0.7258	4.2610	2.0912
144	0.1286	0.2528	2.8705	1.5213
576	0.0615	0.0683	3.7018	1.8882
2304	0.0301	0.0167	4.0889	2.0317

than one for all values of N. Hence, the order of the theoretical estimate of Proposition 16, $t = 1.5$, is sharp.

We close by noting that, even though we have restricted our attention to divergence-free interpolants, the scope of our methods is more general. They may be adapted to construct other classes of customized RBFs, e.g. curl-free interpolants. If $v = 3$, $s = 3$, and if $\mathbf{B}$ consists of $B_1(x) = (0, -x_3, x_2)^T$, $B_2(x) = (x_3, 0, -x_1)^T$, and $B_3(x) = (-x_2, x_1, 0)^T$, where $x = (x_1, x_2, x_3)^T$, then $\mathcal{G}_{\mathbf{B}} = \mathcal{G}_{\mathrm{curl}}$, the admissible space of *irrotational* vector-valued functions. In this case, the general form of an irrotational, 3×3 matrix-valued RBF is given by

$$\Phi_{\mathrm{curl}}(x) := \{\nabla\nabla^T\}\psi(x),$$

with all of its columns being in $\mathcal{G}_{\mathrm{curl}}$, as one can verify directly.

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